

Emergent AdS_3 and BTZ black hole from weakly interacting hot 2d CFT*

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ABSTRACT: We investigate emergent holography of weakly coupled two-dimensional hyperKähler sigma model on $T^*\mathbb{C}\mathbb{P}^{N-1}$ at zero and finite temperature. The sigma model is motivated by the spacetime conformal field theory dual to the near-horizon geometry of Q_1 D1-brane bound to Q_5 D5-brane wrapped on $\mathbb{T}^4 \times \mathbb{S}^1$, where $N = Q_1 Q_5$. The sigma model admits nontrivial instanton for all $N \geq 2$, which serves as a local probe of emergent holographic spacetime. We define emergent geometry of the spacetime as that of instanton moduli space via Hitchin's information metric. At zero temperature, we find that emergent geometry is AdS_3 . At finite temperature, time-periodic instanton is mappable to zero temperature instanton via conformal transformation. Utilizing the transformation, we show that emergent geometry is precisely that of the non-extremal, non-rotating BTZ black hole.

KEYWORDS: Statistical Methods, Black Holes in String Theory, AdS-CFT Correspondence, Sigma Models.

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1. Introduction

In recent work [1], exploiting instantons in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, we were able to extract an emergent holographic dual geometry even at *weak coupling* $g_{\text{YM}}^2 \ll 1$ and *small rank* $N \sim \mathcal{O}(1)$. Specifically, we examined moduli space geometry of the Yang-Mills instantons at finite temperature and found that Hitchin’s information metric [2]¹

$$G_{\text{AB}}^{\text{info}}(Z) \equiv \int_{\mathbb{R}^4} \left\langle \mathcal{L}_{\text{YM}}(\partial_A \log \mathcal{L}_{\text{YM}})(\partial_B \log \mathcal{L}_{\text{YM}}) \right\rangle, \quad (1.1)$$

of the Yang-Mills instanton density \mathcal{L}_{YM} exhibits features identifiable as a Schwarzschild-like black hole whose geometry asymptotes to five-dimensional anti-de sitter space at infinity. Given that this is the regime outside Maldacena’s AdS/CFT correspondence [4], where string worldsheet and quantum effects are violent, the finding in [1] is quite striking and deserves further investigation.

In this work, we continue pursuing the idea of [1] but in a lower dimensional context. Specifically, we consider a class of (4, 4) superconformal field theory, described by two-dimensional nonlinear hyperKähler sigma model whose target space is $T^*\mathbb{CP}^{N-1}$, the cotangent bundle of \mathbb{CP}^{N-1} . For $N \rightarrow \infty$, the sigma model is known to emerge in the infrared limit as holographic dual to the near-horizon geometry of D1-D5 system wrapped on $\mathbb{T}^4 \times \mathbb{S}^1$ with $N = Q_1 Q_5$, where Q_1, Q_5 are the numbers of D1- and D5-branes. This was first suggested in [4] and further studied in [5]–[9].² A natural question then arises: as for the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, is there a holographic dual geometry emergent out of such sigma model at weak coupling and at small N ?

¹For reviews and further discussions on information geometry of instantons, see [3].

²For a recent review on the subject, see [10].

In this work, we shall study geometry of moduli space of holomorphic instantons for the nonlinear sigma model on $T^*\mathbb{CP}^{N-1}$ by exploiting Hitchin's information metric (1.1). We shall find that the geometry is precisely that of the three-dimensional anti-de Sitter space at zero temperature and, at finite temperature, of the non-extremal, non-rotating BTZ black hole! Given that this is the regime where string worldsheet and quantum loop corrections are expected to be large, the result suggests a certain rigidity property of the three-dimensional anti-de Sitter space and the BTZ black hole when they emerge as holographic geometries.

This paper is organized as follows. In section 2, we recall relevant aspects of nonlinear hyperKähler sigma model on target space $T^*\mathbb{CP}^{N-1}$ from the holographic dual of near-horizon geometry of the D1-D5 system. In section 3, we focus on period maps to the two cycles in the base manifold \mathbb{CP}^{N-1} , and propose to study holomorphic instantons of (2,2) supersymmetric nonlinear sigma model on \mathbb{CP}^{N-1} . In section 4, we study geometry of the instanton moduli space at zero temperature. We show that Hitchin's information metric is precisely that of the AdS_3 . In section 5, we study sigma model calorons, viz. periodic instantons at finite temperature with trivial holonomy. The calorons are straightforwardly constructible from the zero-temperature instantons via appropriate conformal mapping. Utilizing the map, we show that Hitchin's information metric is precisely that of non-extremal, non-rotating BTZ black hole. In section 6, we discuss a potentially subtle issue in identifying topology of the 'worldsheet' of the D1-D5 effective string, which is specific to AdS_3 and BTZ black hole.

2. Target space conformal field theory

In this section, we shall recapitulate the microscopic theory of D1-D5 system wrapped on \mathbb{T}^4 [5]-[9]. Here, the D1-branes are "instanton" strings inside the D5-branes. It is well known that the moduli space of Q_1 instantons in $U(Q_5)$ gauge theory on \mathbb{T}^4 is the resolved $\text{Sym}_{Q_1 Q_5}(\tilde{\mathbb{T}}^4)$. In the small instanton size limit, this description reduces to the gauge theory corresponding to the system of Q_1 D1-branes and Q_5 D5-branes. Here, the D1-branes are wrapped on \mathbb{S}^1 and the D5-branes are wrapped on $\mathbb{T}^4 \times \mathbb{S}^1$. The relevant limit is when $\text{Vol}(\mathbb{T}^4) \sim \alpha'^2$, while $\text{Vol}(\mathbb{S}^1) \gg \sqrt{\alpha'}$.

The low-energy dynamics of the above D1-D5 brane complex is described by two-dimensional (4,4) quiver gauge theory of gauge group $U(Q_1) \times U(Q_5)$. By construction, the theory lives on \mathbb{S}^1 . The gauge theory can also have non-zero θ angle corresponding to the relative $U(1)$ gauge group. The field contents arising from massless excitations are as follows. From (1,1) string, there are one 4d $\mathcal{N} = 2$ vector multiplet (A_a, Y_i) and one hypermultiplet Y_m , all in adjoint representation of $U(Q_1)$. From (5,5) string, there are again one 4d vector multiplet (B_a, X_i) and one hypermultiplet X_m , all in adjoint representation of $U(Q_5)$. From (1,5) and (5,1) strings, there arise a hypermultiplet $\chi = (\chi_1, \chi_2)^T$ transforming as a doublet of diagonal $SU(2)_R$ subgroup of internal $SO(4)_I \simeq SU(2) \times SU(2)$ and as a bi-fundamental representation of $U(Q_1) \times U(Q_5)$. Thus, under the relative $U(1)$ gauge group, (χ_1, χ_2) carry charges $(+1, -1)$.

The supersymmetric ground state is determined by two sets of D-flatness conditions for the adjoints of $U(Q_1)$ and $U(Q_5)$, respectively. We also turn on $SU(2)_R$ triplet Fayet-Iliopoulos terms $\zeta \equiv (\zeta, \zeta_c)$ along the $U(1)$ gauge subgroups. Decompose the D-flatness conditions irreducibly into traceless and trace parts. The traceless parts combined with the $SU(Q_1) \times SU(Q_5)$ gauge invariance fixes the two adjoint hypermultiplets X_m and Y_m completely. The trace parts then put constraints solely on bi-fundamental hypermultiplets χ , now deformed by the Fayet-Iliopoulos parameters:

$$\begin{aligned} \chi_1 \otimes \chi_1^* - \chi_2^T \otimes \chi_2^{T*} &= \zeta \\ \chi_1 \otimes \chi_2^T &= \zeta_c. \end{aligned} \tag{2.1}$$

It is now known [11, 12] that the equations (2.1) define the moduli space $T^*\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$, the cotangent bundle of the hyper-Kähler manifold $\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$. More specifically, in (2.1), the first equation for χ_1 describes the base $\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$ manifold, while the second equation for χ_2 defines the cotangent space as the fiber. The Fayet-Iliopoulos parameter ζ is identifiable with the moduli of the hyper-Kähler metric on $T^*\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$. The simplest situation is when $Q_1 Q_5 = 2$, yielding the Eguchi-Hanson space. In the limit ζ is tuned to zero, it approaches the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. Likewise, for $N \equiv Q_1 Q_5 > 2$, the moduli space approaches the symmetric product orbifold:

$$T^*\mathbb{C}\mathbb{P}^N \longrightarrow \text{Sym}_N(\mathbb{C}^2) = \frac{(\mathbb{C}^2)^N}{\mathbb{S}_N}. \tag{2.2}$$

The singularity that appears in this limit corresponds to the cycle of length N of the permutation group \mathbb{S}_N . It is associated with the chiral primary operator with dimension $h = \tilde{h} = (N - 1)/2$. We shall be primarily interested in the moduli subspace associated with the resolution of this cycle. Other singularities in the moduli space corresponding to other cycles are orthogonal to the above one, and hence are inherent to the symmetric product orbifold itself.

Thus, the Higgs branch of the quiver gauge theory corresponds to $U(1)$ gauged linear sigma model on the hyperKähler moduli space $T^*\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$. This theory is expected to flow in the infrared to $(4, 4)$ supersymmetric CFT_2 with central charge $c = \tilde{c} = 6Q_1 Q_5$. In particular, integrating out the $U(1)$ vector multiplet that interacts strongly in the infrared, the theory is reduced to nonlinear sigma model on $T^*\mathbb{C}\mathbb{P}^{Q_1 Q_5 - 1}$ [13].

3. Holomorphic instantons in CFT_2

As discussed in the previous section, Fayet-Iliopoulos deformation resolves the singularity on the Higgs branch and blows it up to a cycle of length N . In this case, the second homology group of the cotangent bundle $T^*\mathbb{C}\mathbb{P}^{N-1}$ is of rank one, generated by a 2-cycle inside the base $\mathbb{C}\mathbb{P}^{N-1}$. Since the CFT_2 at the infrared, the nonlinear sigma model on $T^*\mathbb{C}\mathbb{P}^{N-1}$, lives on \mathbb{S}^1 on which both D1- and D5-branes were wrapped, there now exists instantons that maps $\Sigma_0 = \mathbb{S}^1 \times \mathbb{R}_t \simeq \mathbb{C}$ to the base $\mathbb{C}\mathbb{P}^{N-1}$ at zero temperature or $\Sigma_\beta = \mathbb{S}^1 \times \mathbb{S}_\beta^1 \simeq \mathbb{T}_2$ to $\mathbb{C}\mathbb{P}^{N-1}$ at finite temperature $T = 2\pi/\beta$. They are worldsheet instantons of the effective D1-D5 strings,³ equivalently, holomorphic instantons inside the instanton strings. For example,

³Instantons of this sort were considered from the viewpoint of 6d supergravity in [14].

for the smallest value of $N - 1$, viz. $Q_1 Q_5 = 2$, the target manifold is the Eguchi-Hanson space. The instanton is the well known harmonic map to the two sphere blown up from $\mathbb{C}^2/\mathbb{Z}_2$.

We are primarily interested in understanding geometry of the moduli space of these instantons, framed inside the moduli (sub)space of the Higgs branch. To this end, we shall describe the holomorphic worldsheet instantons effectively as instantons in two-dimensional $\mathbb{C}\mathbb{P}^{N-1}$ model. Evidently, such truncation keeps only the (2,2) supersymmetry manifest, but this does not affect the conclusions we shall be drawing. For $\mathbb{C}\mathbb{P}^{N-1}$, $\dim H^{1,1}(\mathbb{R}) = 1$ and the Kähler class is specified by a single parameter. Correspondingly, the sigma model is specified by a choice of the Kähler potential $K(Z_\alpha, Z_\alpha^*)$, where Z_α, Z_α^* are complex chiral superfields (which just renames χ_1 's). In the Kähler potential, part that is globally defined on $\mathbb{C}\mathbb{P}^{N-1}$ belongs to the D-term, and part whose Kähler form $J = d \wedge \bar{d}K$ shifts complex cohomology classes belongs to the F-term. To construct instanton solutions, we shall excite bosonic part of the $\mathbb{C}\mathbb{P}^{N-1}$ model, which consists of N complex scalar fields Z_α subject to the constraint $|Z_\alpha| = 1$. It is given by

$$S = \frac{N}{\lambda^2} \int_{\Sigma} \left[|\partial_m Z_\alpha|^2 + \frac{1}{4} (Z_\alpha^* \partial_m Z_\alpha - Z_\alpha \partial_m Z_\alpha^*)^2 \right]. \tag{3.1}$$

Here, λ is a dimensionless coupling constant. This is a unique action involving two derivatives and retaining a local U(1) invariance $Z_\alpha(x) \rightarrow e^{i\varepsilon(x)} Z_\alpha(x)$. Alternatively, the model can be defined without the constraint $|Z| = 1$ but with manifest U(1) gauge invariance by introducing a Lagrange multiplier μ^2 and an abelian gauge potential A_m , respectively. This amounts to defining the $\mathbb{C}\mathbb{P}^{N-1}$ sigma model in terms of gauged (2, 2) linear sigma model, which constructs $\mathbb{C}\mathbb{P}^{N-1}$ as a quotient of \mathbb{C}^N . In this formulation, bosonic part of the action reads

$$S = \frac{N}{\lambda^2} \int_{\Sigma} \left[|D_m Z_\alpha|^2 - \mu^2 (|Z_\alpha|^2 - 1) \right], \tag{3.2}$$

where $D_m Z_\alpha \equiv (\partial_m + iA_m) Z_\alpha$. The equations of motion are

$$(D_m^2 + \mu^2) Z_\alpha = 0, \quad |Z_\alpha|^2 = 1, \quad (Z_\alpha^* D_m Z_\alpha - Z_\alpha D_m Z_\alpha^*) = 0. \tag{3.3}$$

Solving the latter two equations, we find

$$A_m = \frac{i}{2} (Z_\alpha^* \partial_m Z_\alpha - Z_\alpha \partial_m Z_\alpha^*). \tag{3.4}$$

The BPS equation is derivable by rewriting the action under the condition $|Z_\alpha|^2 = 1$ as

$$S = \frac{N}{\lambda^2} \int_{\Sigma} \left[|(D_m \mp i\epsilon_{mn} D_n) Z_\alpha|^2 + 2\pi \mathcal{F} \right] \geq \frac{2\pi N}{\lambda^2} Q, \tag{3.5}$$

where \mathcal{F} is the instanton charge density and

$$Q \equiv \int_{\Sigma} \mathcal{F} = -\frac{i}{2\pi} \int_{\Sigma} \epsilon_{mn} (D_m Z_\alpha)^* (D_n Z_\alpha) = \frac{1}{4\pi} \int_{\Sigma} \epsilon_{mn} F_{mn} \in \mathbb{Z} \tag{3.6}$$

is the instanton charge. It is integrally quantized U(1) gauge flux. From (3.5), the BPS equation follows in the form:

$$D_m Z_\alpha = \pm i \epsilon_{mn} D_n Z_\alpha . \tag{3.7}$$

Thus, in complex coordinates, components of an instanton configuration are holomorphic sections of a line bundle over Σ . It also follows from (3.7) that the Lagrange multiplier is set by the instanton charge density

$$\mu^2 = -\frac{1}{2} \epsilon_{mn} F_{mn} . \tag{3.8}$$

The BPS instantons that solve (3.7) are well known [15] - [17]. It is well known that the dimension of the moduli space for charge k instantons in \mathbb{CP}^{N-1} is given by

$$\dim_{\mathbb{C}} \mathcal{M}_k^N = kN = k Q_1 Q_5 . \tag{3.9}$$

We shall now construct holomorphic instantons explicitly in a suitable parametrization suited for our purpose, both at zero and finite temperature. We shall then study geometry of the instanton moduli space by computing Hitchin's information metric.

4. Emergent AdS₃ geometry at zero temperature

Consider first the holomorphic instantons at zero temperature. The 'worldsheet' of the effective D1-D5 string is $\Sigma_0 = \mathbb{S}^1 \times \mathbb{R}_t$, which is conformally equivalent to \mathbb{C} . Introduce complex coordinates on the worldsheet $z \equiv (x^1 + ix^2)$, $\bar{z} \equiv (x^1 - ix^2)$. The most general holomorphic instanton solution of (3.7) topological charge 1 is given by

$$Z_\alpha = \frac{w_\alpha}{|w|}, \quad \text{where} \quad w_\alpha = (z - z_o)u_\alpha + \rho_\alpha, \quad u_\alpha^* \rho_\alpha = 0, \quad |u|^2 = 1. \tag{4.1}$$

Here, moduli parameters of the holomorphic instanton are $z_o = x_o^1 + ix_o^2$, $\bar{z}_o = x_o^1 - ix_o^2$ for the center, $\rho = \sqrt{|\rho|^2}$ for the size, and ρ_α/ρ for the SU(N) orientation, respectively. The u_α parametrizes the SU(N) vacuum. With appropriate SU(N) rotation, we can choose

$$u_\alpha = \delta_{\alpha,1} \quad \text{and} \quad \rho_\alpha = \rho \delta_{\alpha,2} \tag{4.2}$$

while satisfying the conditions in (4.1). Thus, there are one complex modulus parameter z_o and one real modulus parameter ρ , specifying center and size of the instanton, respectively. They range over $z_o \in \mathbb{C}$ and $\rho \in \mathbb{R}^+$. We are primarily interested in the three-dimensional subspace in the moduli space \mathcal{M}_k^N in (3.9) — this is the subspace orthogonal to the SU(N) orientation.

The Lagrangian density of the holomorphic instanton reads

$$\mathcal{F}(z; z_o) = \frac{1}{4\pi} \epsilon^{mn} F_{mn}(z; z_o). \tag{4.3}$$

Substituting the instanton solution (4.1), we find that

$$\mathcal{F}(z, z_o) = \frac{1}{\pi} \frac{|\rho|^2}{(|z - z_o|^2 + |\rho|^2)^2}. \tag{4.4}$$

Motivated by Hitchin’s proposal [2], we propose holographic geometry in terms of *quantum-averaged* information metric of the instanton moduli space as

$$G_{\text{AB}}^{\text{info}}(z_o) = \int_{\Sigma} dz d\bar{z} \left\langle \mathcal{F}(\partial_A \log \mathcal{F})(\partial_B \log \mathcal{F}) \right\rangle. \tag{4.5}$$

Here, the bracket refers to normalized path integral over the fields (Z_α, Z_α^*) . Semiclassically, saddle-point configuration dominates (4.5). By elementary computations, as was done in [18], the information geometry spans three-dimensional subspace of ρ and z_o . On this space, the information metric (4.5) is given by

$$(ds)_{\text{instanton}}^2 = \frac{R^2}{\rho^2} \left[d\rho^2 + dz_o d\bar{z}_o \right], \tag{4.6}$$

where $R^2 = 4/3$. Had we considered an instanton of topological charge k instead, the metric remains the same as (4.6), except that $R^2 \rightarrow R^2 k^2$.

We see that, as probed by the holomorphic instanton, the three-dimensional Euclidean anti-de Sitter space emerges as a holographic geometry of two-dimensional (2,2) supersymmetric sigma model over \mathbb{CP}^{N-1} , which spans relevant part of the (4,4) hypermultiplet nonlinear sigma model over the hyperKähler moduli space $T^*\mathbb{CP}^{N-1}$. We also see that the emergent AdS geometry (4.6) does not depend on the rank $N = Q_1 Q_5$ and the sigma model coupling λ , the feature shared by the information geometry of instantons in four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory.

5. Emergent BTZ black hole geometry at finite temperature

Consider next the holomorphic instanton at finite temperature $T = 2\pi/\beta$. They are sigma model calorons. Adopting the reasoning of [19], we expect that calorons with *trivial* holonomy would dominate the thermal partition function at semiclassical level. In Matsubara formulation, the ‘worldsheet’ of the effective D1-D5 string is $\Sigma_\beta = \mathbb{S}^1 \times \mathbb{S}_\beta^1$. For reasons we shall return later, we will open up \mathbb{S}^1 to \mathbb{R}^1 of the ‘worldsheet’ and consider $\Sigma_\beta = \mathbb{R} \times \mathbb{S}_\beta^1$. The ‘worldsheet’ coordinates $y = y_1 + iy_2$ covers Σ_β with $y_2 \simeq y_2 + \beta$ identified with compact coordinate on \mathbb{S}_β^1 . With this choice, the most general unit charge caloron of trivial holonomy is given by [20, 21]

$$Z_\alpha = \frac{w_\alpha}{|w|}, \quad \text{where} \quad w_\alpha = u_\alpha e^{\frac{2\pi}{\beta}(y-y_o)} + v_\alpha, \quad |u|^2 = |v|^2 = 1, \quad \arg(u_\alpha^* v_\alpha) = 0. \tag{5.1}$$

As before, u_α parametrizes $SU(N)$ vacuum. Now, in contrast to the instanton at zero temperature, v_α is not directly interpretable as the size parameter ρ . This is because u_α and v_α are no longer $SU(N)$ orthogonal — the condition $\arg(u_\alpha^* v_\alpha) = 0$ still leaves $\text{Re}(u_\alpha^* v_\alpha)$ arbitrary — and because rescaling of v_α is equivalent to shifting y_0 . By appropriate $SU(N)$ rotation, we choose to parametrize the moduli as

$$u_\alpha = \delta_{\alpha,1} \quad \text{and} \quad v_1 = \sqrt{1-a^2}, \quad |v_2| = a, \quad v_3 = \dots v_N = 0 \quad (0 \leq a \leq 1). \tag{5.2}$$

This choice is compatible with the conditions on caloron’s moduli parameters given in (5.1).

In fact, the caloron with trivial holonomy can be related to the instanton at zero temperature. This is most conveniently seen by making the conformal transformation

$$\exp\left(\frac{2\pi}{\beta}y\right) = \frac{2\pi}{\beta}z. \tag{5.3}$$

With suitable overall rescaling which leaves Z_α intact, we can now rewrite the caloron (5.1) in the form of the zero temperature instanton (4.1):

$$w_\alpha = (z - z_o)u_\alpha + \rho_\alpha \tag{5.4}$$

by judiciously arranging the moduli parameters so that they satisfy the constraints (4.1). However, this does not mean that the caloron is the same as the instanton at zero temperature. Rather, it means that the caloron moduli parameters (y_0, a) are reinterpretable in terms of the zero temperature instanton moduli parameters:

$$\rho_\alpha = \frac{\beta}{2\pi}e^{\frac{2\pi}{\beta}y_o}[v_\alpha - (u^* \cdot v)u_\alpha], \quad \rho = a\frac{\beta}{2\pi}\left|e^{\frac{2\pi}{\beta}y_o}\right|, \quad z_o = -\frac{\beta}{2\pi}\sqrt{1 - a^2}e^{\frac{2\pi}{\beta}y_o}. \tag{5.5}$$

It indicates that the calorons, once conformally mapped, constitute a proper subset of all possible instantons at zero temperature.

Now, the complex coordinate z covers the entire complex plane because of the periodicity of Euclidean time, $y_2 \simeq y_2 + \beta$. Thus, we can utilize the previous computation of holographic geometry for the zero temperature instanton to extract information metric of the caloron. In other words, interpreted in terms of the instanton moduli (ρ, z_o) , Hitchin's information metric for the caloron is the AdS₃ space in the Poincaré coordinates. But then, the information metric of the caloron expressed in terms of the caloron moduli (a, y_o) is obtainable by substituting the relations (5.5). This yields

$$\begin{aligned} (ds)_{\text{caloron}}^2 &= \frac{R^2}{\rho^2} [d\rho^2 + dz_o d\bar{z}_o] \\ &= R^2 \left[\frac{da^2}{a^2(1-a^2)} + \left(\frac{2\pi}{\beta}\right)^2 \left(\frac{dy_1^2}{a^2} + \left(\frac{1}{a^2} - 1\right) dy_2^2 \right) \right], \end{aligned} \tag{5.6}$$

where $y_o \rightarrow y_1 + iy_2, \bar{y}_o \rightarrow y_1 - iy_2$. Redefining the coordinates as $a = 1/r$, we find that

$$(ds)_{\text{caloron}}^2 = R^2 \left[\frac{dr^2}{r^2 - 1} + \left(\frac{2\pi}{\beta}\right)^2 \left(r^2 dy_1^2 + (r^2 - 1) dy_2^2 \right) \right]. \tag{5.7}$$

Here, because $a \in (0, 1]$, the 'radial' variable r ranges only over $r \in [1, \infty)$. We then recognize that the moduli space metric (5.7) is precisely the metric of the non-extremal, non-rotating BTZ black hole, whose horizon is located at $r_+ = 1$.

Notice that the emergent geometry does not depend on the rank $N = Q_1 Q_5$ of the hypermultiplet sigma model and the coupling λ at all. This is in contrast to the situation of four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills theory, where the emergent geometry at finite temperature certainly deviated from the AdS₅ Schwarzschild black hole. It suggests that there exists certain rigidity or nonrenormalization property against string worldsheet and quantum loop corrections.

6. Topology

There remains an issue concerning global aspect of the holographic geometry, . In constructing instanton and caloron of the hypermultiplet sigma model as holomorphic worldsheet instantons of the D1-D5 effective string, we have tacitly taken the ‘worldsheet’ to have topology of $\Sigma_0 \simeq \mathbb{C}$ and $\Sigma_\beta \simeq \mathbb{R} \times \mathbb{S}_\beta^1$. Thus, the instanton at zero temperature is a localized lump and the caloron at finite temperature is a periodic array around the Euclidean time direction.

On the other hand, by construction, the D1-D5 effective string theory is defined on \mathbb{S}^1 , not on \mathbb{R} . That would mean that we should have constructed instantons at zero temperature as a periodic array around \mathbb{S}^1 and calorons at finite temperature as a doubly period array around $\mathbb{S}^1 \times \mathbb{S}_\beta^1$. Stated differently, calorons with single or double periodicity are the relevant configurations in so far as the boundary condition of hypermultiplets on the ‘worldsheet’ is concerned. If this were the correct identification of the worldsheet topology, a technical issue pertains since, according to the theorem of [22], there is no $k = 1$ instantons on \mathbb{T}_2 for any value of N , a feature shared with Yang-Mills theories on \mathbb{T}^4 [23]. On the other hand, the instanton exists on $\mathbb{S}^1 \times \mathbb{R}_t$. Change of the worldsheet topology arose from turning on finite temperature. Since this should be a smooth process, we think the $k = 1$ instanton ought to exist not only on $\mathbb{S}^1 \times \mathbb{R}_t$ but also on \mathbb{T}_2 .

A possible resolution would be the well-known string theoretic mechanism that the relative U(1) gauge subgroup is coupled to NS-NS or R-R two-form potential, and they contribute (generically non-integral) U(1) flux background on the ‘worldsheet’ of the D1-D5 effective string. Combined with the instanton or caloron configuration, the gauge-invariant net U(1) flux can be arranged to zero, and evade conditions of the aforementioned no-go theorem. In such a situation, the topology of the ‘worldsheet’ can be taken consistent with the D1-D5 effective string theory, viz. $\Sigma_0 = \mathbb{S}^1 \times \mathbb{R}$ or $\Sigma_\beta = \mathbb{S}^1 \times \mathbb{S}_\beta^1$, where the size of \mathbb{S} is taken large compared to the string scale. With such topology, the emergent geometries described by the metrics (4.6), (5.7) indeed describe correctly the global AdS₃ and BTZ black holes.

An alternative is to take the correct ‘worldsheet’ topology as $\Sigma_0 = \mathbb{C}$ at zero temperature and as $\Sigma_\beta = \mathbb{R} \times \mathbb{S}_\beta^1$ at finite temperature. In this case, we evade the no-go theorem and have smooth $k = 1$ instanton interpolation with the temperature. But then, the emergent geometry (5.7) should be interpreted as describing the BTZ black hole in extreme high temperature limit only.

We intend to report clarification of this point in a separate work.

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References

- [1] S.J. Rey and Y. Hikida, *5D black hole as emergent geometry of weakly interacting 4D hot Yang-Mills gas*, [hep-th/0507082](#).
- [2] N.J. Hitchin, *The geometry and topology of moduli spaces*, Lecture Notes in Mathematics **1451**, Springer, Heidelberg, 1988, pp. 1-48.
- [3] S. Amari, M.K. Murray and J.M. Rice, *Statistics and differential geometry*, Monographs on Statistics and Applied Probability **48**, Chapman and Hall, London, 1993;
D. Groisser and M.K. Murray, *Instantons and the information metric*, *Ann. Glob. Ann. Geom.* **15** (1997) 519-537.
- [4] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)].
- [5] A. Giveon, D. Kutasov and N. Seiberg, *Comments on string theory on AdS_3* , *Adv. Theor. Math. Phys.* **2** (1998) 733 [[hep-th/9806194](#)].
- [6] J. de Boer, H. Ooguri, H. Robins and J. Tannenhauser, *String theory on AdS_3* , *JHEP* **12** (1998) 026 [[hep-th/9812046](#)].
- [7] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of $(2,0)$ superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [[hep-th/9712117](#)].
- [8] R. Dijkgraaf, *Instanton strings and hyperkaehler geometry*, *Nucl. Phys.* **B 543** (1999) 545 [[hep-th/9810210](#)].
- [9] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3* , *JHEP* **04** (1999) 008 [[hep-th/9903219](#)].
- [10] J.R. David, G. Mandal and S.R. Wadia, *Microscopic formulation of black holes in string theory*, *Phys. Rept.* **369** (2002) 549 [[hep-th/0203048](#)].
- [11] N. Seiberg and E. Witten, *The $D1/D5$ system and singular CFT*, *JHEP* **04** (1999) 017 [[hep-th/9903224](#)].
- [12] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [[hep-th/9908142](#)].
- [13] N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, *Hyperkähler metrics and supersymmetry*, *Commun. Math. Phys.* **108** (1987) 535.
- [14] A. Mikhailov, *$D1D5$ system and noncommutative geometry*, *Nucl. Phys.* **B 584** (2000) 545 [[hep-th/9910126](#)].
- [15] D.J. Gross, *Meron configurations in the two-dimensional $O(3)$ sigma model*, *Nucl. Phys.* **B 132** (1978) 439.
- [16] E. Witten, *Instantons, the quark model, and the $1/N$ expansion*, *Nucl. Phys.* **B 149** (1979) 285.
- [17] A. D'Adda, P. Di Vecchia and M. Luscher, *Confinement and chiral symmetry breaking in CP^{N-1} models with quarks*, *Nucl. Phys.* **B 152** (1979) 125.
- [18] S. Yahikozawa, *The information metric on instanton moduli spaces in nonlinear sigma models*, *Phys. Rev.* **D 69** (2004) 026122 [[physics/0307131](#)].

- [19] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43.
- [20] I. Affleck, *The role of instantons in scale invariant gauge theories*, *Nucl. Phys.* **B 162** (1980) 461; *The role of instantons in scale invariant gauge theories, 2. The short distance limit*, *Nucl. Phys.* **B 171** (1980) 420.
- [21] A. Actor, *Temperature dependence of the CP^{N-1} model and the analogy with quantum chromodynamics*, *Fortschr. Phys.* **33** (1985) 333.
- [22] J.L. Richard and A. Rouet, *The CP^1 model on the torus: contribution of instantons*, *Nucl. Phys.* **B 211** (1983) 447;
P. van Baal, *Nahm gauge fields for the torus*, *Phys. Lett.* **B 48** (1999) 26.
- [23] S. Mukai, Nagoya, *Math. J.* **81** (1981) 153;
P.J. Braam and P. van Baal, *Comm. Math. Phys.* **122** (1989) 122;
H. Schenk, *On a generalized fourier transform of instantons over flat tori*, *Commun. Math. Phys.* **116** (1988) 177.